Survival of a snake in an hostile environment

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Abstract. We consider a Brownian snake subjected to killing in an inhomogeneous medium. We show an equation satisfied by the probability of survival. Conditionally on survival, the Brownian snake has the law of a snake with a drift that we specify. This study include "path-dependent killings". In the case of constant drift, we give a large deviation principle and derive some estimates for the caracteristics of this process. The results can also be stated in terms of super-Brownian motion and super-Brownian motion with drift. Then, we show how to annihilate a (general non-positive) drift by adding immigration. The conditioning to non-extinction of super-Brownian motion with constant non-positive drift is also studied.

1. Introduction

The Brownian snake introduced by Le Gall is widely recognized to be a convenient representation of the super-Brownian motion as well as a powerful tool for the applications to certain pde or particle systems. We refer the reader to [Lg] for a comprehensive treatement of the subject but, for the convenience of the reader, we will recall some basic facts in the next section which should suffice for the reading of this paper. The notations used in the present introduction and the rest of the paper are also detailed in that section. The reader which is not very familiar to the subject could find convenient to read first next section before the following introduction.

In the present paper we consider a Brownian snake (W_s) living in a partially absorbing medium. This snake is absorbed (we could say killed in a more conventional way) at a rate given by the non-negative measurable function V. Conditionally on (W_s), the probability of survival up to time τ is given by

$$\mathrm{P}(\text{survival}|(W_s)) = \exp\left(-\int_0^\tau V(W_r)\,dr\right).$$

Different models corresponding to various times τ could a priori be considered : exit time of a domain, fixed time, but, in order to use the connections with super-Brownian motion, we will study here the case $\tau = \tau_1$ (hitting time of 1 by the local time at 0 of the lifetime of the Brownian snake). For some results given in the next sections the function $V\ : \mathcal{W} \to \mathbb{R}_+$ can be

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fairly general but in the present introduction we restrict ourselves to the case $V(w) = V(\hat{w})$. The probability of survival, up to time τ_1 of a Brownian snake starting from $\tilde{x}$, killed according to V, is then given by

$$\mathbb{E}_{\tilde{x}}\left[\exp\left(-\int_0^{\tau_1} V(\hat{W}_s)\,ds\right)\right]. \tag{1}$$

This probability is positive as soon as V is bounded on a neighbourhood of x. From now on, we suppose that V is locally bounded and we set

$$h(x) = -\log \mathbb{E}_{\tilde{x}}\left[\exp\left(-\int_0^{\tau_1} V(\hat{W}_s)\,ds\right)\right].$$

This function is solution on whole $\mathbb{R}^d$ of the following p.d.e :

$$\frac{\Delta}{2}h - 2\,h^2 + V = 0. \tag{2}$$

We have supposed here and in the rest of the introduction, that the "spatial motion" of the Brownian snake is simply Brownian motion in $\mathbb{R}^d$, hence the presence of its generator $\Delta/2$. This p.d.e satisfied by h can be found in the litterature on super-Brownian motion. Indeed, it is well known that a super-Brownian motion (Y_t) can be constructed from (W_s); we recall it briefly in the Formula (11) of the next section. Therefrom, $h(x)$ can be rewritten using the formula :

$$\int_0^{+\infty} Y_t(V)\,dt = \int_0^{\tau_1} V(\hat{W}_s)\,ds. \tag{3}$$

Here we extend (2) to more general cases for V i.e. V depending on the whole trajectory and we give a proof using arguments on Brownian snake. See section 3.2.

Since the survival event, as specified earlier, is of positive probability, conditioning by this event is straightforward. We can thus define the law $\mathbb{P}_{w_0}^{(V,\tau_1)}$ of the Brownian snake starting from w_0 and conditioned to survive up to time τ_1 the killing at rate V and stopped at that time, from the law $\mathbb{P}_{w_0}$ of the Brownian snake starting from w_0, by the following formula (where ϕ is an appropriate test function) :

$$\mathbb{E}_{w_0}^{(V,\tau_1)}\left[\phi(W_{s\wedge\tau_1}, s \geq 0)\right] = \frac{\mathbb{E}_{w_0}\left[\phi(W_{s\wedge\tau_1}, s \geq 0)\ \exp\left(-\int_0^{\tau_1} V(\hat{W}_s)\,ds\right)\right]}{\mathbb{E}_{w_0}\left[\exp\left(-\int_0^{\tau_1} V(\hat{W}_s)\,ds\right)\right]}. \tag{4}$$

Of course we can re-formulate this law as the one of a super-Brownian motion penalized by the weight function :

$$\exp\left(h(x) - \int_0^{+\infty} Y_t(V)\,dt\right). \tag{5}$$

It happens that the law defined by (4) is known. It is the law of a b–snake, as named in [AS]. Roughly speaking, a b–snake is obtained from the ordinary Brownian snake by addition of a drift term; in particular its lifetime is a reflecting Brownian motion with drift $-b(W_s)\,ds$. The relevant drift function b for our conditioned snake is simply $b(w) = 2\,h(\hat{w})$ as stated in Proposition 8. Note that in the case where $V(\cdot)$ is constant equal to V, the function $h(\cdot)$ is also constant, equal to $h = \sqrt{V/2}$ as shows Equation (2). In that case the law of the snake conditioned to survival is easily described : its lifetime is a reflecting Brownian motion with drift $-\sqrt{2\,V}$, stopped at τ_1, and the conditional law knowing the lifetime is the same as ordinary Brownian snake.

Proposition 8 could also be stated in terms of super-Brownian motion or even historical Brownian motion when $V(\cdot)$ is no more of the type $V(w) = V(\hat{w})$. It says that the law of the superprocess obtained from super-Brownian motion by the penalization given in (5) is a super-Brownian motion with drift b given previously. Super-Brownian motion with drift has for instance been shown to be the limit of rescaled contact processes ([DP]) or rescaled Lotka-Volterra competing species models ([CP]); in both case the drift is even constant.

A limit case of our study is when $V \to +\infty$ on a domain U of $\mathbb{R}^d$ and equals 0 elsewhere. We get Brownian snake (or super-Brownian motion) conditioned not to reach U. In the recent paper [LgW], Le Gall and Weill provide a deep study of the one-dimensional Brownian snake conditioned to stay positive, motivated by problems on asymptotics for planar maps. We briefly discuss at the end of section 4 this particular case.

We want to described the behaviour of the snake conditioned to survive i.e. the b-snake or equivalently the behaviour of a super-Brownian motion with drift. In particular, what is the probability of exit from a "big ball" i.e. the tail distribution of

$$R = \sup_s |\hat{W}_s| = \inf\{\rho; \forall t,\ Y_t(\{x, |x| > \rho\}) = 0\} \ ? \tag{6}$$

For (non-conditioned) Brownian snake, or equivalently for the support of super-Brownian motion (Y_t), the answer is well known, as a result of excursion theory and a scaling argument. Using the excursion measure whose definition is recalled in section 2, we get

$$\begin{aligned} \mathbb{P}_{\tilde{0}}[R > r] &= 1 - \exp -\mathbb{N}_0[R > r] \\ &= 1 - \exp -\frac{c_1}{r^2} \text{ where } c_1 = \mathbb{N}_0\left[\sup_{s\leq\sigma} |\hat{W}_s| > 1\right] \\ &\sim \frac{c_1}{r^2} \text{ when } r \to +\infty. \end{aligned}$$

For a snake conditioned to survival/a super-Brownian motion with drift, the behaviour is much different. For instance if V is constant (and still starting from 0), excursion representation (see Proposition 6) and scaling lead to

$$\begin{aligned}
\mathbb{P}_{\tilde{0}}^{(V,\tau_1)}[R > r] &= 1 - \exp -\mathbb{N}_0 \left[\mathbf{1}_{\{R>r\}} \exp - \left(\int_0^\sigma V(\hat{W}_s)\, ds \right) \right] \\
&\sim \mathbb{N}_0 \left[1_{\{R>r\}} e^{-V\sigma} \right] \\
&= \int_0^{+\infty} \frac{d\sigma}{\sqrt{2\pi}\, \sigma^{3/2}}\, e^{-V\sigma}\, \mathbb{N}_0^\sigma (R > r) \\
&= \int_0^{+\infty} \frac{d\sigma}{\sqrt{2\pi}\sigma^{3/2}}\, e^{-V\sigma}\, \mathbb{N}_0^1 \left(R > \frac{r}{\sigma^{1/4}} \right) \qquad (7)
\end{aligned}$$

where $\mathbb{N}_0^\sigma$ denotes the law of the Brownian snake excursion with lifetime excursion length σ. But the asymptotics of the above probability of exit is known by [DZ] (or [S1]). Using $\approx$ to mean "same exponential speed", we have

$$\mathbb{N}_0^1(R > A) \approx \exp \left(-\frac{3}{2} A^{4/3} \right) .$$

Then the integral above should behave like

$$\int_0^{+\infty} \frac{d\sigma}{\sqrt{2\pi}\sigma^{3/2}}\, e^{-V\sigma}\, e^{-\frac{3}{2}\, \frac{r^{4/3}}{\sigma^{1/3}}} \approx e^{-2^{5/4}\, V^{1/4}\, r} \qquad (8)$$

where the last estimate is obtained by minimizing the negative exponential exponent.

This result may seem strange : since V is constant, why does the surviving snake has much less probability to go far than the ordinary snake, when we condition by the fact that killing has not occured ? Because excursions with bigger length are less probable. For instance, still in the case of constant killing at rate V, the added lengths of excursions, which sum up to τ_1, is distributed according to

$$\begin{aligned}
\mathbb{E}_{\tilde{0}}^{(V,\tau_1)} \left(e^{-\lambda\, \tau_1} \right) &= \frac{\mathbb{E}_{\tilde{0}} \left(e^{-\lambda\, \tau_1}\, e^{-V\, \tau_1} \right)}{\mathbb{E}_{\tilde{0}} \left(e^{-V\, \tau_1} \right)} \\
&= \exp - \frac{1}{\sqrt{2}} \left(\sqrt{V + \lambda} - \sqrt{V} \right) . \qquad (9)
\end{aligned}$$

We note in particular that, a contrario to the unconditioned case, τ_1 has finite expectation $1/\sqrt{8\, V}$. If $V(\cdot)$ is not constant but bounded from below by V, it is easy to see that (9) now becomes an upper bound. See Proposition 7 for a formalisation of a monotonicity remark, also applying for instance to the quantity R considered before.

We want to make rigorously the above calculation (8) and extend it by stating a large deviation principle for the law of a rescaling of (W_s). Such a large deviation principle has been proved in [S1] for a (non-conditioned) Brownian snake when the lifetime process is a normalized Brownian excursion (that is, under $\mathbb{N}_0^1$) and when it is spatially rescaled by the factor $r^{-3/4}$. For our present purpose we first need to state an analogous result under the excursion measure $\mathbb{N}_0$. As an outline (otherwise see Theorem 10 below) this large deviation principle has speed r and rate function

$$J(W) = \frac{1}{2}\int_0^\sigma \dot\zeta_s^2\, ds + \frac{1}{4}\int_0^\sigma \frac{|\dot{\hat{W}}_s|^2}{|\dot\zeta_s|}\, ds.$$

Then we come back to the large deviations of a Brownian snake surviving the killing at a constant rate V. We are able to state a large deviation result still with speed r but this time the spatial normalization factor is $1/r$, which confirms the rough estimate (8). The rate function is now $W \to J(W) + V\ \sigma(W)$, see Theorem 11 for the precise statement. As an application we compute a large deviation estimate for R as hinted in (8) and also for the extinction time

$$H = \sup_{s\le\tau_1} \zeta_s = \inf\{t > 0,\ Y_t = 0\}$$

for which we obtain the result :

$$\lim_{r\uparrow+\infty} \frac{1}{r}\log \mathbb{P}_{\hat{0}}^{(V,\tau_1)}(H > r) = -2^{3/2}\ V^{1/2}.$$

See Proposition 12 and 13 for these derivations.

In the following section 8, we are interested in a way to transform a Brownian snake/Super-Brownian motion with (non-positive) drift into another one, but without drift. In terms of super-Brownian motion, this can be done by adding to the original super-Brownian motion with drift a superposition of independent "super-Brownian excursions" with random starting time and random starting point, both distributed according to a certain Poisson measure. See Theorem 15 of section 8 for the precise statement. This can be viewed also in the following terms: super-Brownian motion is a super-Brownian motion with (non-positive) drift to which we add a certain immigration.

Finally the last section concentrates on the super-Brownian motion with constant non-positive drift conditioned to non-extinction. This conditioning for super-Brownian motion is classical, see for instance [Ev] or [Ov]. It can be constructed by conditioning the extinction time to be greater than h and then let $h \uparrow +\infty$. We first remark that we obtain a similar limit if we condition the total mass Z of super-Brownian motion to be L and let L tend to $+\infty$. However this becomes false in the case of super-Brownian motion with drift. The conditioning by infinite total mass leads to the same limit as in the case without drift whereas the conditioning to infinite extinction time leads to another law that we specify.

2. Some background on the Brownian snake

In this paper we will use the following notations :
$\mathbb{R}_+ = [0, +\infty)$.
c denotes a constant whose value is unimportant and may change from line to line.

$\mu(\phi)$: integral of function ϕ with respect to measure μ.
$\mathrm{Supp}(\mu)$: support of the measure μ.
A^C : complement of the set A.
$\mathbf{1}_A$: indicator function of set A.
$|x|$: for $x \in \mathbb{R}^d$, euclidean norm of x.
$A \wedge B$: infimum of the numbers A and B.
$\mathcal{C}(X,Y)$: set of continuous functions from metric space X to metric space Y.
$\mathcal{M}_F(X)$: set of finite measures on the metric space X, equipped with the topology of weak convergence and its Borel σ-algebra.
$\mathcal{K}_d$: set of compact subsets of $\mathbb{R}^d$ equipped with the Hausdorff metric.

This paper deals with processes taking their values in the space $\mathcal{W}$ of stopped paths. A stopped path is a couple (w,ζ), where $\zeta \geq 0$ is called the lifetime of the path, and $w : \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous mapping which is constant on $[\zeta, +\infty)$. We often write w for (w,ζ), and $\zeta(w)$ for the lifetime. The distance on $\mathcal{W}$ is $d(w,w') = \sup_{t\geq 0} |w(t)-w'(t)|+|\zeta(w)-\zeta(w')|$, making $\mathcal{W}$ a Polish space. We denote by

1. $\hat{w} = w(\zeta)$ the endpoint of w,
2. $w_{\leq r}$ or $w^{\leq r}$ the path of lifetime $\zeta(w) \wedge r$ such that, for $u \geq 0$, $w_{\leq r}(u) = w(u \wedge r)$,
3. $\tilde{x}$ the path of lifetime 0 started at $x \in \mathbb{R}^d$,
4. $w_1 \oplus w_2$ the concatenation of two paths w_1 and w_2 such that $\hat{w}_1 = w_2(0)$, defined as the path of lifetime $\zeta_1 + \zeta_2$ given by :

$$w_1 \oplus w_2(u) = w_1(u)\ \mathbf{1}_{\{u \leq \zeta_1\}} + w_2(u-\zeta_1)\ \mathbf{1}_{\{\zeta_1 < u \leq \zeta_1+\zeta_2\}}.$$

Let us fix a diffusion with values in $\mathbb{R}^d$, and denote by A its generator. A simple example of process in $\mathcal{W}$ is the so-called A–path process $(\tilde{W}_s)_{s\geq 0}$. It is a Markov process such that $\zeta(\tilde{W}_s) = \zeta(\tilde{W}_0) + s$ and $\tilde{W}_s$ has the law of $\tilde{W}_0 \oplus \xi$ where ξ is a trajectory of the A–diffusion starting from $\hat{\tilde{W}}_0$ and stopped at time s. We will denote $\tilde{\mathrm{P}}_w$ its law when it starts from w and L its generator.

The Brownian snake starting at $w \in \mathcal{W}$, with spatial motion governed by A, whose law will be denoted $\mathbb{P}_w$, is the strong Markov continuous process $(W_s, s \geq 0)$ with values in $\mathcal{W}$, characterized by the following properties:

1. $W_0 = w$;
2. The lifetime process $\zeta_s = \zeta(W_s)$ is a reflecting Brownian motion in $\mathbb{R}_+$ starting at $\zeta(w)$;
3. The conditional distribution $\mathbb{P}_w^{[\zeta]}$ of $(W_s, s \geq 0)$ knowing $(\zeta_s, s \geq 0)$ is that of an inhomogeneous Markov process whose transition kernels are described as follows : for every $s < s'$,
   - $W_{s'}^{\leq m} = W_s^{\leq m}$ for $m = \inf_{r\in[s,s']} \zeta_r$; this property will be called "snake property" in the sequel;

- $(W_{s'}(m+t), 0 \le t \le \zeta_{s'} - m)$ is independent of W_s conditionally on $W_s(m)$ and has the law of a diffusion in $\mathbb{R}^d$ with generator A, starting from $W_s(m)$ and stopped at time $\zeta_{s'} - m$.

From the Brownian snake starting at $\tilde{x}$, $x \in \mathbb{R}^d$, one can construct the $\mathcal{M}_F(\mathcal{W})$–valued process $(H_t)_{t\ge 0}$ called historical Brownian motion starting at $\delta_{\tilde{x}}$ by :

$$H_t = \int_0^{\tau_1} d_{(s)} L_s^t \;\; \delta_{W_s} \tag{10}$$

where L_s^t is the local time of the lifetime process $(\zeta_.)$ at level t and time s, and $\tau_1 = \inf\left\{s \ge 0; L_s^0 > 1\right\}$ is the hitting time of 1 for the local time at level 0. Super-Brownian motion is then the $\mathcal{M}_F(\mathbb{R}^d)$–valued process $(Y_t)_{t\ge 0}$ given as the image measure of H_t by the mapping $w \to \hat{w}$ so that

$$Y_t = \int_0^{\tau_1} d_{(s)} L_s^t \;\; \delta_{\hat{W}_s} \;. \tag{11}$$

To simplify notations we have chosen a normalisation which differs from the usual one by a factor 4 and our super-Brownian motion is usually called the $(A, 4\,z^2, 0)$–super-process, and identically for historical Brownian motion. This representation formula (11) gives, via the occupation time formula, the Equation (3) stated in the introduction. It can of course be reinforced in :

$$\int_0^{+\infty} H_t(\phi)\; dt = \int_0^{\tau_1} \phi(W_s)\; ds \tag{12}$$

for a measurable $\phi : \mathcal{W} \to \mathbb{R}_+$. The so-called "range" of super-Brownian motion amounts to be

$$S(W) = \{\hat{W}_s,\; s \le \tau_1\}. \tag{13}$$

As $\tilde{x}$ is a regular point for the recurrent process (W_s), it is possible to define the associated excursion measure $\mathbb{N}_x$. To be more specific and introduce some notation useful later, let $(\alpha_i, \beta_i)_{i\in I}$ be the excursion intervals of (ζ_s) out of 0, up to time τ_1 and $(W^i)_{i\in I}$ be the corresponding "snake excursions" that is $W_s^i = W_{(\alpha_i+s)\wedge\beta_i}$. Under $\mathbb{P}_{\tilde{x}}$, the random point measure

$$\sum_{i\in I} \delta_{(L^0_{\alpha_i}, W^i)}(dl\, dW) \tag{14}$$

is a Poisson measure with intensity $\mathbf{1}_{(0,1)}(l)\, dl\; \mathbb{N}_x(dW)$. Under $\mathbb{N}_x$, the law of the lifetime is distributed as the Itô measure of positive excursions of Brownian motion and the conditional law knowing the lifetime ζ is $\mathbb{P}_{\tilde{x}}^{[\zeta]}$ as before. We will also use the probability $\mathbb{N}_x^\sigma$ which is the conditioning of $\mathbb{N}_x$ knowing that the lifetime excursion has duration σ.

Under $\mathbb{N}_0$ and when spatial motion is Brownian ($A = \Delta/2$), the following scaling identity holds :

$$\mathbb{N}_0[F(W)] = \frac{1}{\sqrt{\alpha}}\, \mathbb{N}_0\left[F\left(\theta_\alpha(W)\right)\right] \tag{15}$$

where $F:\ \mathcal{C}(\mathbb{R}_+,\mathcal{W})\to\mathbb{R}_+$ is a measurable test function, $\alpha>0$ and θ_α is the following scaling operator :

$$\theta_\alpha(W)_s(u)=\left(\alpha^{1/4}W_{\frac{s}{\alpha}}(\frac{u}{\alpha^{1/2}}),\alpha^{1/2}\zeta_{\frac{s}{\alpha}}\right). \tag{16}$$

Spatial motion governed by A being fixed, we recall that the notion of b-snake can be defined for instance by a well-posed martingale problem, see [AS]. An equivalent definition, at least when b is bounded, is to say that the law $\mathbb{P}^b_w$ of the b–snake starting from $w\in\mathcal{W}$ has a density with respect to the law $\mathbb{P}_w$ of the ordinary Brownian snake when we restrict to events prior to time s and this density is :

$$d_s=\exp\left(-\int_0^s b(W_r)\,d\beta_r-\frac{1}{2}\int_0^s b^2(W_r)\,dr\right) \tag{17}$$

where (β_s) is the martingale part of the semi-martingale (ζ_s), given by Tanaka's formula. Supposing for simplicity that $b(w)=\hat{b}(\hat{w})$, the super-process associated via Equation (11) to a b–snake starting from $\tilde{x}$ is the $(A,4z^2,-\hat{b})$–super-process starting from δ_x that we will call super-Brownian motion with drift $-\hat{b}$ i.e. the solution of the martingale problem

$$(\hat{b}\text{–MP})\begin{cases} Y_0=\delta_x\\ \forall\phi\in\mathcal{D}(A),\quad Y_t(\phi)=Y_0(\phi)+\int_0^t Y_s(A\phi)\,ds-\int_0^t Y_s(\hat{b}\,\phi)\,ds+M_t(\phi)\\ \text{where the local martingale } M \text{ has quadratic variation}\\ \langle M(\phi)\rangle_t=4\int_0^t Y_s(\phi^2)\,ds.\end{cases}$$

Analogous statement with general b is given for the historical process in [DS].

3. Probability of survival

3.1. **Sufficent conditions of survival.**

Proposition 1. *The probability of survival up to time τ_1 of a Brownian snake, starting from $\tilde{x}_0$ and killed according to the measurable non-negative function $V:\mathcal{W}\to\mathbb{R}_+$, is given by :*

$$\mathbb{E}_{\tilde{x}_0}\left[\exp\left(-\int_0^{\tau_1}V(W_s)\,ds\right)\right]=\exp -h(x_0) \tag{18}$$

where $h\ :\ \mathbb{R}^d\to\mathbb{R}_+$ equals

$$h(x_0)=\mathbb{N}_{x_0}\left[1-\exp\left(-\int_0^\sigma V(W_s)\,ds\right)\right]. \tag{19}$$

For such a survival to occur with positive probability, each of the following condition is sufficient

- *$V(\cdot)$ is bounded on a ball B in $\mathcal{W}$ centered at $\tilde{x}_0$.*
- *V satisfies, for B a ball in $\mathcal{W}$ centered at $\tilde{x}_0$,*

$$\int_0^{+\infty}\tilde{\mathrm{E}}_{\tilde{x}_0}\left(V(\tilde{W}_t)\,\mathbf{1}_{\{\tilde{W}_t\subset B\}}\right)\,dt\ <+\infty.$$

Proof. Equations (18,19) come from the usual exponential formula of excursion theory, where σ denotes, under $\mathbb{N}_{x_0}$, the length of the lifetime excursion. The probability of survival is positive if and only if $h(x_0)$ is finite. By discussing whether or not the Brownian snake exits a ball B in $\mathcal{W}$ centered at $\tilde{x}_0$, we get the following upper bound :

$$h(x_0) \leq \mathbb{N}_{x_0}\left[\left(1-e^{-\int_0^\sigma V(W_s)\,ds}\right)\mathbf{1}_{\{\forall s,\, W_s\subset B\}}\right] + \mathbb{N}_{x_0}\left[\{W_s, s\in[0,\sigma]\}\not\subset B\right].$$

The second term is finite; otherwise we obtain a contradiction with the continuity of $s \to W_s$. To treat the first one, let us suppose first that $V(\cdot)$ is bounded by V_{max} on B. By using the "law" of σ under $\mathbb{N}_{x_0}$ we get :

$$\begin{aligned}\mathbb{N}_{x_0}\left[\left(1-e^{-\int_0^\sigma V(W_s)\,ds}\right)\mathbf{1}_{\{\forall s,\, W_s\subset B\}}\right] &\leq \mathbb{N}_{x_0}\left[1-e^{-\sigma V_{max}}\right] \\ &= \int_0^{+\infty}\frac{d\sigma}{\sqrt{2\pi}\,\sigma^{3/2}}\left(1-e^{-\sigma V_{max}}\right) < +\infty.\end{aligned}$$

To obtain the second sufficient condition of survival we simply use $1-e^{-x}\leq x$ for $x\in\mathbb{R}_+$ to get

$$\begin{aligned}\mathbb{N}_{x_0}\left[\left(1-e^{-\int_0^\sigma V(W_s)\,ds}\right)\mathbf{1}_{\{\forall s,\, W_s\subset B\}}\right] &\leq \mathbb{N}_{x_0}\left[\int_0^\sigma V(W_s)\,\mathbf{1}_{\{W_s\subset B\}}\,ds\right] \\ &= \int_0^{+\infty}\tilde{\mathrm{E}}_{\tilde{x}_0}\left[V(\tilde{W}_t)\,\mathbf{1}_{\{\tilde{W}_t\subset B\}}\right]\,dt.\end{aligned}$$

The last equality follows from Bismut's description of the Brownian excursion (see [RY] p. 502), because, under the measure $\mathbb{N}_{x_0}(dW)\,\mathbf{1}_{[0,\sigma]}(ds)$, ζ_s is distributed as the Lebesgue measure on $\mathbb{R}_+$ and conditionally on $\zeta_s = t$, W_s is an A-Markov process in $\mathbb{R}^d$ starting from x_0 and stopped at time t.

Note that this second sufficient condition of survival can be realized even if V is not bounded in a neighbourhood of $\tilde{x}_0$. For instance for $V(w) = \hat{V}(\hat{w})$, the above condition amounts to the finiteness of

$$\int_B G(x_0,z)\,\hat{V}(z)\,dz$$

where G stands for the Green function associated to the A–Markov process and B is now a ball in $\mathbb{R}^d$ centered at x_0. For example, if $\hat{V}$ is moreover radial i.e. $\hat{V}(x) = \overline{V}(|x|)$, $x_0 = 0$ and spatial motion is Brownian, the condition reduces to

$$\int_0 r^{2-d}\,\overline{V}(r)\,r^{d-1}\,dr = \int_0 r\,\overline{V}(r)\,dr < +\infty.$$

3.2. **Equation for the survival probability.** Coming back to the case of a general $V(\cdot)$, we define an extension $b\,:\,\mathcal{W}\to\mathbb{R}_+$ of $2\,h$ defined before, in the sense that $b(\tilde{x}) = 2\,h(x)$ and obtain an equation satisfied by this function on $\mathcal{W}$.

Proposition 2. *For $V : \mathcal{W} \to \mathbb{R}_+$ measurable and locally bounded i.e. bounded on a neighbourhood of every $w \in \mathcal{W}$, we set*

$$b(w) = 2\, \mathbb{N}_{\hat{w}} \left[1 - \exp\left(- \int_0^\sigma V(w \oplus W_s)\, ds \right) \right]. \tag{20}$$

This function b satisfies the following equation for every $w \in \mathcal{W}$:

$$b(w) = 2 \int_0^{+\infty} dt \;\; \tilde{P}_t^b(V)(w) \tag{21}$$

where $(\tilde{P}_t^b)$ is the semi-group associated to the A-path process killed according to b. For every $w \in \mathcal{W}$, the probability of survival up to time τ_1 of a Brownian snake starting at w and killed according to V is

$$\mathbb{E}_w \left[\exp\left(- \int_0^{\tau_1} V(W_s)\, ds \right) \right] = \exp\left(- \frac{b(\widetilde{w(0)})}{2} - \int_0^\zeta b(w_{\leq r})\, dr \right). \tag{22}$$

In particular, for every $x \in \mathbb{R}^d$, $2\, h(x) = b(\tilde{x})$.

Proof. We write

$$\begin{aligned}
\frac{1}{2}\, b(w) &= \mathbb{N}_{\hat{w}} \left[\int_0^\sigma \exp\left(- \int_0^s V(w \oplus W_u)\, du \right) \; V(w \oplus W_s)\, ds \right] \\
&= \mathbb{N}_{\hat{w}} \left[\int_0^\sigma \exp\left(- \sum_i \int_{\alpha_i}^{\beta_i} V(w \oplus W_s^{\leq \zeta_{\alpha_i}} \oplus W_u^i)\, du \right) \; V(w \oplus W_s)\, ds \right].
\end{aligned}$$

where the (α_i, β_i) are, at fixed time s, the excursion intervals of the lifetime above its future infimum, up to time s and on such an excursion interval, $W_u^i(v) = W_u(\zeta_{\alpha_i} + v)$ for $u \in (\alpha_i, \beta_i)$ defines the corresponding "small snakes". But, by Bismut's description of the Brownian excursion, we know that, under the measure $\mathbf{1}_{[0,\sigma]}(ds)\, \mathbb{N}_x(dW)$, ζ_s is distributed as the Lebesgue measure on $\mathbb{R}_+$; conditionally on $\zeta_s = t$, W_s is an A-Markov process in $\mathbb{R}^d$ starting from x and stopped at time t and the lifetime $(\zeta_u, u \leq s)$ is a three dimensional Bessel process run until it last hits t. By Pitman's Theorem the excursions of a three dimensional Bessel process above its future infimum are distributed as the excursions of a reflecting Brownian motion out of 0. Thanks to this and the description of the conditional law of the Brownian snake knowing its lifetime, the random measure

$$\sum_i \delta_{(\zeta_{\alpha_i}, W^i_{(\alpha_i + \cdot) \wedge \beta_i})}(du, dW)$$

is, under the measure $\mathbf{1}_{[0,\sigma]}(ds)\, \mathbb{N}_x(dW)$ and conditionally on W_s of lifetime t, a Poisson measure with intensity

$$\mathbf{1}_{[0,t]}(du) \;\; 2\, \mathbb{N}_{W_s(u)}(dW).$$

Thus the exponential formula shows that, under the previous conditional law, the expectation of

$$\exp\left(-\sum_i \int_{\alpha_i}^{\beta_i} V(w \oplus W_s^{\leq \zeta_{\alpha_i}} \oplus W_u^i)\, du\right)$$

is

$$\begin{aligned}\exp - \int_0^t du\, 2\, \mathbb{N}_{W_s(u)} \left[1 - \exp\left(-\int_0^\sigma V(w \oplus W_s^{\leq u} \oplus W_v')\, dv\right)\right] \\ = \exp - \int_0^t du\, b(w \oplus W_s^{\leq u}).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\frac{1}{2}\, b(w) &= \tilde{\mathrm{E}}_w \left[\int_0^{+\infty} dt\, V(\tilde{W}_t)\, \exp - \int_0^t du\, b(\tilde{W}_u)\right] \\ &= \int_0^{+\infty} dt\;\; \tilde{P}_t^b(V)(w)\end{aligned}$$

which is the desired result.

To prove (22) we introduce the time T_0 of return to 0 of the lifetime; we apply the Markov property at T_0, as recalled in (29) below; on $[0, T_0]$, we decompose according to the excursions above the minimum as in [Lg] V.Lemma 5; on $[T_0, \tau_1]$, we decompose according to the excursions out of the trivial path $\widetilde{w(0)}$. This argument leads without difficulty to Formula (22).

Proposition 3. *When V is continuous on $\mathcal{W}$, the function b given by (20) belongs to the domain of L and satisfies on $\mathcal{W}$ the equation*

$$Lb - b^2 + 2\,V = 0. \tag{23}$$

Proof. Equation (21) gives :

$$\frac{1}{h}\, \left(\tilde{P}_h^b(b) - b\right)(w) = -\frac{2}{h} \int_0^h dt\;\; \tilde{P}_t^b(V)(w).$$

By the Feller property of the (killed) A–path, the right-hand side has the limit $-2\,V(w)$ as $h \downarrow 0$ so b belongs to the domain of the A-path process killed according to b whose generator is $(L - b)$. Hence $(L - b)\,b = -2\,V$ which is the announced formula (23).

3.3. **Limit or special cases.**

Corollary 4. *If $V(w) = +\infty\, \mathbf{1}_U(w)$ for an open subset U of $\mathcal{W}$, we have*

$$b(w) = 2\, \mathbb{N}_{\hat{w}}\left(\{w \oplus W_s;\; s \in (0, \sigma)\} \cap U \neq \emptyset\right) \tag{24}$$

and, for every w in the interior of U^C, $Lb(w) = b^2(w)$.

Proof. Formula (24) for b is straightforward. For $n \in \mathbb{N}^*$, we set

$$V_n(w) = n\, \inf\left(n \inf_{u \notin U} d(w, u), 1\right)$$

so that $V_n(w) \uparrow V(w)$ for every $w \in \mathcal{W}$. If b_n is associated to V_n, Beppo-Levi Theorem implies $b_n(w) \uparrow b(w)$ for every $w \in \mathcal{W}$. Moreover V_n is continuous (even Lipschitz) so that b_n belongs to the domain of L and $Lb_n = b_n^2 - 2\,V_n$. But for $w \notin U$, $V_n(w) = 0$. If w is in the interior of U^C, we have $Lb_n(\cdot) = b_n^2(\cdot)$ on a ball B centered at w. This implies by the martingale problem associated with the A-path process $(\tilde{W}_s)$ starting from w, that for every stopping time T lower than the exit time of B,

$$\tilde{\mathrm{E}}_w \left[b_n(\tilde{W}_T) - b_n(w) - \int_0^T b_n^2(\tilde{W}_r)\, dr \right] = 0.$$

By Beppo-Levi Theorem we can pass to the limit and obtain the same equation where b replaces b_n. This implies that $b(\tilde{W}_s) - b(w) - \int_0^s b^2(\tilde{W}_r)\, dr$ is a martingale, up to the exit time of B. We conclude that b belongs to the domain of L and $Lb(w) = b^2(w)$.

Corollary 5. *If $V(w) = \hat{V}(\hat{w})$ where $\hat{V}$ is continuous on $\mathbb{R}^d$, the probability of survival of a Brownian snake starting from $x \in \mathbb{R}^d$ is equal to* $\exp -h(x)$ *where $h : \mathbb{R}^d \to \mathbb{R}_+$ belongs to the domain of A and satisfies on whole $\mathbb{R}^d$ the p.d.e.*

$$A\,h - 2\,h^2 + \hat{V} = 0. \tag{25}$$

If $\hat{V}$ is moreover with compact support, h is the unique non-negative solution vanishing at infinity on $\mathbb{R}^d$ of the previous equation.

Proof. Equation (25) follows readily from Equation (23). It is clear from the definition (19) that if $\hat{V}$ has compact support, h tends to 0 at infinity. We mean that h is less than any arbitrary small value if we look at it outside a certain compact set of $\mathbb{R}^d$. Consider now another non-negative solution $\tilde{h}$ of (25) vanishing at infinity. If $h - \tilde{h}$ takes somewhere a positive value, it admits a positive maximum at a point x_1. We obtain a contradiction by using the maximum principle for the generator A :

$$0 \geq A(h - \tilde{h})(x_1) = 2h^2(x_1) - 2\tilde{h}^2(x_1) > 0.$$

We deduce $h - \tilde{h} \leq 0$ and by symmetry, $h - \tilde{h} = 0$.

For example, in the case of a continuous radial V, that is $V(x) = V(|x|)$ and with Brownian spatial motion, the function h is also radial : $h(x) = h(|x|)$ and the expression of the Laplacian in spherical coordinates leads to

$$h''(r) + \frac{d-1}{r} h'(r) - 4\,h^2(r) + 2V = 0. \tag{26}$$

Unfortunately it is not possible in general to solve this equation given a function V but it is at least possible to know which V correspond to a function h. In the limiting case $V(r) = +\infty\ \mathbf{1}_{\{r \geq A\}}$ and $d = 1$, we obtain that $h'' = 4\,h^2$ on $(0, A)$, $h(A-) = +\infty$. A classical computation gives, for $r \in (0, A)$,

$$h(r) = \frac{3\,G(+\infty)^2}{8\,A^2}\,G^{-1}\left(G(+\infty)\,\frac{r}{A}\right) \tag{27}$$

where

$$G(y) = \int_1^y \frac{dz}{\sqrt{z^3-1}} \text{ and } G(+\infty) = \frac{2\sqrt{\pi}\,\Gamma(7/6)}{\Gamma(2/3)}.$$

In particular the probability for a one dimensional super-Brownian motion starting at 0 to stay in $(-A, A)$ is

$$\exp -h(0) = \exp -\left(\frac{3\pi}{2}\left(\frac{\Gamma(7/6)}{\Gamma(2/3)}\right)^2 \frac{1}{A^2}\right) \approx \exp -\frac{2.21}{A^2}.$$

4. Law of a surviving snake

4.1. **Markov property.** The law $\mathbb{P}_w^{(V,\tau_1)}$ of the Brownian snake starting from w and conditioned to survive up to time τ_1 the killing at rate V, and stopped at that time, has a density with respect to the law $\mathbb{P}_w$ of the Brownian snake starting from w and stopped at time τ_1, given, on the $\sigma-$algebra of events prior to time τ_1, by

$$D_w = \frac{d\mathbb{P}_w^{(V,\tau_1)}}{d\mathbb{P}_w} = \frac{\exp\left(-\int_0^{\tau_1} V(W_s)\,ds\right)}{\mathbb{E}_w\left[\exp\left(-\int_0^{\tau_1} V(W_s)\,ds\right)\right]}$$

and, in particular, when $w = \tilde{x},\ x \in \mathbb{R}^d$,

$$D_{\tilde{x}} = \exp\left(\frac{b(\tilde{x})}{2} - \int_0^{\tau_1} V(W_s)\,ds\right). \tag{28}$$

The Markov property of the Brownian snake can be written in our context, with appropriate test functions F, G, as :

$$\begin{aligned} &\mathbb{E}_w\left[F(W_{r\wedge\tau_1},\ r \le s)\ G(W_{(s+u)\wedge\tau_1},\ u \ge 0)\right] \\ &= \mathbb{E}_w\left[F(W_{r\wedge\tau_1},\ r \le s)\ \int \mathbb{P}_{W_{s\wedge\tau_1}}(dW')\ \ G(W'_{u\wedge\tau_{1-L_s^0}}, u \ge 0)\right]. \end{aligned} \tag{29}$$

It is easy to deduce a Markov property for (L_s^0, W_s) in the conditioned context that we write :

$$\begin{aligned} &\mathbb{E}_w^{(V,\tau_1)}\left[F(W_{r\wedge\tau_1},\ r \le s)\, G(W_{(s+u)\wedge\tau_1},\ u \ge 0)\right] \\ = \ &\mathbb{E}_w^{(V,\tau_1)}\left[F(W_{r\wedge\tau_1},\ r \le s)\ \int \mathbb{P}_{W_{s\wedge\tau_1}}^{(V,\tau_{1-L_s^0})}(dW')\ \ G(W'_{u\wedge\tau_{1-L_s^0}}, u \ge 0)\right]. \end{aligned}$$

From now on, $\mathbb{P}_w^{(V,\tau_1)}$ and $\mathbb{E}_w^{(V,\tau_1)}$ will be abbreviated into $\mathbb{P}_w^{(V)}$ and $\mathbb{E}_w^{(V)}$.

4.2. **Excursions.** We consider the excursions out of a trivial path $\tilde{x}$, $x \in \mathbb{R}^d$, keeping the notations of (14).

Proposition 6. *Under $\mathbb{P}^{(V)}_{\tilde{x}}$, the random measure*

$$\sum_{i \in I} \delta_{(L^0_{\alpha_i}, W^i)}(dl\, dW)$$

is a Poisson measure with intensity $\mathbf{1}_{(0,1)}(l)\, dl\, \mathbb{N}^{(V)}_x(dW)$ where $\mathbb{N}^{(V)}_x(dW)$ is the infinite measure on $\mathcal{C}(\mathbb{R}_+, \mathcal{W})$ given, for a measurable $F:\, \mathcal{C}(\mathbb{R}_+, \mathcal{W}) \to \mathbb{R}_+$, by

$$\mathbb{N}^{(V)}_x[F] = \mathbb{N}_x \left[\, F(W)\; \exp - \left(\int_0^\sigma V(W_s)\, ds \right) \right].$$

Proof. It suffices to show that, for F as above,

(30)

$$\mathbb{E}^{(V)}_{\tilde{x}} \left[\exp - \sum_{i \in I} F(L^0_{\alpha_i}, W^i) \right] = \exp - \left[\int_0^1 dl \;\int \mathbb{N}_x(dW) \left(1 - e^{-F(l,W)} \right) \;\; e^{-\int_0^\sigma V(W_s)\, ds} \right].$$

We rewrite the left-hand side using the definition of $\mathbb{E}^{(V)}_{\tilde{x}}$, noting that

$$\int_0^{\tau_1} V(W_s)\; ds = \sum_{i \in I} \int_0^{\beta_i - \alpha_i} V(W^i_s)\; ds.$$

We obtain a ratio where the exponential formula can be applied to both numerator and denominator :

$$\frac{\mathbb{E}_{\tilde{x}} \left[\exp - \sum_{i \in I} \left(F(L^0_{\alpha_i}, W^i) + \int_0^{\beta_i - \alpha_i} V(W^i_s)\; ds \right) \right]}{\mathbb{E}_{\tilde{x}} \left[\exp - \sum_{i \in I} \int_0^{\beta_i - \alpha_i} V(W^i_s)\; ds \right]}$$

$$= \quad \frac{\exp - \int_0^1 dl \int \mathbb{N}_x(dW) \left(1 - e^{-F(l,W) - \int_0^\sigma V(W_s)\, ds} \right)}{\exp - \int_0^1 dl \int \mathbb{N}_x(dW) \left(1 - e^{-\int_0^\sigma V(W_s)\, ds} \right)}.$$

This last expression reduces to the right-hand side of (30), completing the proof.

This excursion representation implies a straightforward corollary on the monotonicity of certain variables with respect to V, which was less clear on the definition of $\mathbb{P}^{(V)}_w$. We say that a non-negative variable U decreases with V if $V_1(\cdot) \le V_2(\cdot)$ implies $\mathbb{E}^{(V_1)}_{\tilde{x}}(\varphi(U)) \ge \mathbb{E}^{(V_2)}_{\tilde{x}}(\varphi(U))$ for every $x \in \mathbb{R}^d$ and every non-decreasing $\varphi:\, \mathbb{R}_+ \to \mathbb{R}_+$.

Proposition 7. *For every measurable non-negative F, every variable of the form (with the notations of (30))*

$$(i)\; U = \sum_{i \in I} F(L^0_{\alpha_i}, W^i) \;\; or \;\; (ii)\; U = \sup_{i \in I} F(L^0_{\alpha_i}, W^i)$$

decreases with V. Examples are :

$$\tau_q = \inf\{s,\ L^0_s > q\}\ (q \in (0,1]),\quad R = \sup_{s\le\tau_1} |\hat{W}_s|,\quad H = \sup_{s\le\tau_1} \zeta_s.$$

Proof. It suffices to check the criteria for $\varphi(\cdot) = 1 - e^{-\lambda\,\cdot}$, $\lambda \ge 0$ in case (i) and $\varphi(\cdot) = \mathbf{1}_{(A,+\infty)}(\cdot)$, $A \ge 0$ in case (ii). In case (i) we read the result on (30). The example τ_q is obtained for $F(\alpha, W) = \mathbf{1}_{\{\alpha\le q\}}\ \sigma(W)$. In case (ii), we use the definition of Poisson measures to get

$$\begin{aligned}
\mathbb{P}^{(V)}_{\tilde{x}}(U > A) &= \mathbb{P}^{(V)}_{\tilde{x}}(\exists i \in I,\ F((L^0_{\alpha_i}, W^i) > A) \\
&= 1 - \exp - \int_0^1 dl\ \mathbb{N}_{\tilde{x}} \left[\mathbf{1}_{\{F(l,W)>A\}}\, e^{-\int_0^\sigma V(W_s)\, ds}\right]
\end{aligned}$$

and the result follows, with the examples of R and H.

4.3. **Main result.** We now identify the law of the conditioned Brownian snake as the law of a b–snake.

Proposition 8. *Let V be a bounded continuous non-negative function on $\mathcal{W}$. The Brownian snake starting from $\tilde{x}_0$ and conditioned to survive the killing at rate V, up to time τ_1 and stopped at that time, is a b-snake starting from $\tilde{x}_0$ and stopped at time τ_1 where b is given by (20).*

Proof. Note that the boundedness of V implies the boundedness of b and the law of the b–snake is defined by Formula (17). We apply Ito's formula for the Brownian snake with respect to the function

$$F(w) = \int_0^\zeta b(w_{\le r})\, dr$$

as given in [DS] Theorem 2. We obtain

$$F(W_{s\wedge\tau_1}) = F(W_0) + \int_0^{s\wedge\tau_1} b(W_r)\, d\zeta_r + \frac{1}{2}\int_0^{s\wedge\tau_1} Lb(W_r)\, dr.$$

We recall Tanaka's Formula $\zeta_s = \beta_s + (1/2)L^0_s(\zeta)$ where β is a Brownian motion; we note that obviously $F(W_0) = 0$ and $Lb - b^2 + 2\,V = 0$ as proved earlier. We get

$$\begin{aligned}
F(W_{s\wedge\tau_1}) &= \int_0^{s\wedge\tau_1} b(W_r)\, d\beta_r + \frac{b(\tilde{x}_0)}{2}\, L^0_{s\wedge\tau_1} \\
&\quad + \frac{1}{2}\int_0^{s\wedge\tau_1} b^2(\hat{W}_r)\, dr - \int_0^{s\wedge\tau_1} V(\hat{W}_r)\, dr. \qquad (31)
\end{aligned}$$

But the density of $\mathbb{P}_{\tilde{x_0}}^{(V)}$ with respect to $\mathbb{P}_{\tilde{x_0}}$, restricted to $\mathcal{F}_s = \sigma(W_r,\ r \leq s \wedge \tau_1)$, is

$$\begin{aligned} D_{\tilde{x}_0,s} &= \mathbb{E}_{\tilde{x}_0}\left[\exp\left(\frac{b(\tilde{x_0})}{2} - \int_0^{\tau_1} V(W_r)\, dr\right) |\mathcal{F}_s\right] \\ &= \exp\left(\frac{b(\tilde{x_0})}{2} - \int_0^{s\wedge\tau_1} V(W_r)\, dr\right) \\ &\quad \times\ \mathbb{E}_{W_{s\wedge\tau_1}}\left[\exp\left(-\int_0^{\tau_1 - L_s^0} V(W_r')\, dr\right)\right] \end{aligned}$$

where the latter expression is obtained by the Markov property as stated in (29). A slight generalization of Formula (22) is

$$\mathbb{E}_{W_{s\wedge\tau_1}}\left[\exp\left(-\int_0^{\tau_1 - L_s^0} V(W_r')\, dr\right)\right] = \exp\left(-\frac{b(\tilde{x_0})}{2}(1 - L^0_{s\wedge\tau_1}) - F(W_{s\wedge\tau_1})\right)$$

so that we obtain finally :

$$D_{\tilde{x}_0,s} = \exp\left(-\int_0^{s\wedge\tau_1} V(W_r)\, dr + \frac{b(\tilde{x_0})}{2} L^0_{s\wedge\tau_1} - F(W_{s\wedge\tau_1})\right). \tag{32}$$

This can be re-expressed using (31) as

$$D_{\tilde{x}_0,s} = \exp\left(-\int_0^{s\wedge\tau_1} b(W_r)\, d\beta_r - \frac{1}{2}\int_0^{s\wedge\tau_1} b^2(W_r)\, dr\right)$$

which is precisely the density of the law of the b-snake when restricted to $\mathcal{F}_s$.
Remark 1. Formula (32) shows the difficulty to describe the conditional law of $(W_r)_{r\leq s}$ knowing the lifetime. Generally speaking this conditional law is given by the formula

$$\mathbb{E}_{\tilde{x}_0}^{(V)}\left[\phi(W_r,\, r \leq s\wedge\tau_1)|\zeta\right] = \frac{\mathbb{E}_{\tilde{x}_0}\left[\phi(W_r,\, r \leq s\wedge\tau_1)\, D_{\tilde{x}_0,s}|\zeta\right]}{\mathbb{E}_{\tilde{x}_0}\left[D_{\tilde{x}_0,s}|\zeta\right]}. \tag{33}$$

We are not able to give a more explicit description in the general case.
Remark 2. Proposition 8 can of course be re-formulated in the following way : the law of super-Brownian motion penalized according to (5) is the law of super-Brownian motion with drift. A proof can be given without Brownian snake in the case $V(w) = \hat{V}(\hat{w})$ by using Dawson's stochastic calculus for measure valued processes ([Da] chapter 7), once the solution h of (25) is known.

4.4. **Special cases.** Let us examine briefly special cases where the b-snake is more easily described.

If $V(\cdot)$ is constant then $D_{\tilde{x}_0,s}$ is measurable with respect to (ζ_s) and it is clear on Formula (33) that the conditional law given the lifetime of the surviving snake is the same as the one of standard Brownian snake.

If $V(w) = +\infty\ \mathbf{1}_U(\hat{w})$ for a smooth open subset U of $\mathbb{R}^d$, the Brownian snake starting from $\tilde{x},;x \notin U$ and killed according to V can be viewed as the Brownian snake conditioned not to hit U. It is a b–snake with b given by

$$b(w) = 2\,h(\hat{w}) \text{ and } h(x) = \mathbb{N}_x\left(\{\hat{W}_s;\ s \in [0,\sigma]\} \cap U \neq \emptyset\right).$$

When U is smooth, $h(x) \to +\infty$ if $x \to \partial U$. Adapting Formula (32) to this case, we rewrite Formula (33) under the (little) more explicit form :

$$\mathbb{E}^{(V)}_{\tilde{x}}\left[\phi(W_r,\, r \leq s \wedge \tau_1) | \zeta\right] = \frac{\mathbb{E}_{\tilde{x}}\left[\phi(W_r,\, r \leq s \wedge \tau_1)\,\mathbf{1}_{\{\forall r \leq s\wedge\tau_1,\, \hat{W}_r \notin U\}}\ \exp - \left(\int_0^{\zeta_{s\wedge\tau_1}} 2\,h(W_{s\wedge\tau_1}(r))\,dr\right) | \zeta\right]}{\mathbb{E}_{\tilde{x}}\left[\mathbf{1}_{\{\forall r \leq s\wedge\tau_1,\, \hat{W}_r \notin U\}}\ \exp - \left(\int_0^{\zeta_{s\wedge\tau_1}} 2\,h(W_{s\wedge\tau_1}(r))\,dr\right) | \zeta\right]}.$$

Another special case of our study happens when $V(w) = v(\zeta)$. In this case the function b depends on ζ only and more generally spatial motion is irrelevant with respect to this conditioning. We simply obtain the fact that a (linear) Brownian motion (ζ_s), reflecting at 0 and killed according to v has, when conditioned to survive up to τ_1, the law of a (reflecting) Brownian motion with drift $-b$ where b is a solution of the Ricatti equation : $b' - b^2 + 2v = 0$. In particular when conditioned to stay in $[0, A]$, corresponding to $v = +\infty\,\mathbf{1}_{(A,+\infty)}$, the drift is $b(x) = -1/(A - x)$.

Let us conclude this section with particular one dimensional cases. First the law of the Brownian snake conditioned to stay between $-A$ and A up to time τ_1 is a b–snake where $b(w) = 2\,h(|\hat{w}|)$ and h is given by Formula (27). Let us now consider, for $\varepsilon > 0$, the law of the (one-dimensional) Brownian snake sarting at $\tilde{0}$ and conditioned not to hit $(-\infty, -\varepsilon)$ up to τ_1. By the previous Theorem, it is a b_ε-snake with $b_\varepsilon(w) = 2\,h_\varepsilon(|\hat{w}|)$ and h_ε is now the solution of $h''_\varepsilon = 4\,h^2_\varepsilon$ on $(-\varepsilon, +\infty)$ and $h_\varepsilon(-\varepsilon+) = +\infty$. Moreover h_ε has the following "explicit" expression that we transform by scaling :

$$\begin{aligned} h_\varepsilon(x) &= \mathbb{N}_x\left(\{W_s;\ s \in (0,\sigma)\} \cap (-\infty, -\varepsilon) \neq \emptyset\right) \\ &= \frac{1}{(x+\varepsilon)^2}\mathbb{N}_0\left(\{W_s;\ s \in (0,\sigma)\} \cap (-\infty, -1) \neq \emptyset\right) \end{aligned}$$

But in order that h_ε solves the above differential equation, the quantity on the right-hand side of the last line must be $3/2$, so that we conclude $b_\varepsilon(w) = 3/(\hat{w} + \varepsilon)^2$. In [LgW], it is shown that the law of the considered conditioned Brownian snake converges when $\varepsilon \downarrow 0$. Following our description we would be led to a Brownian snake with drift $b(w) = 3/2\hat{w}^2$. However there is a problem to define this process starting from 0 because the drift is unbounded at 0 and

$$\int_0 \left(\frac{3}{2\,\hat{W}_s^2}\right)^2 ds = +\infty$$

showing that Formula (17) cannot be extended to this situation. The divergence of the above martingale is for instance a consequence of the law of the iterated logarithm for the Brownian snake ([S2]).

5. Large deviations of the Brownian snake excursions

In [S1] a large deviation principle has been given for

$$W^r = \left(\frac{1}{r^{3/4}} W_s(\sqrt{r}\,\cdot), \frac{1}{\sqrt{r}}\zeta_s \right)_{s\geq 0}$$

when $r \to +\infty$ where (W_s) is a Brownian snake starting from $\tilde{0}$ whose lifetime is a normalized Brownian excursion, that is, under $\mathbb{N}_0^1$ and with Brownian spatial motion ($A = \Delta/2$). This large deviation principle has speed r and good rate function $\tilde{J}$ given by

$$\tilde{J}(W) = \frac{1}{2}\int_0^1 \dot{\zeta}_s^2\, ds + \frac{1}{4}\int_0^1 \frac{|\dot{\hat{W}}_s|^2}{|\dot{\zeta}_s|} ds$$

when W belongs to the appropriate function space. For the purpose of the present paper we need to adapt this result to the case of a Brownian snake under its excursion measure $\mathbb{N}_0$. We first define some notations.

We denote by $\mathcal{P}$ the space of all $(W_s,\zeta_s)_{s\in[0,\sigma]}$ where $\sigma > 0$ and W is a path-valued function with lifetime ζ, having the snake property, ζ being an excursion of length σ denoted $\sigma(\zeta)$ or even $\sigma(W)$. For every $\sigma_0 > 0$, the subset $\mathcal{P}^{\sigma_0}$ of $\mathcal{P}$ is defined by restricting to the snakes $(W,\zeta) \in \mathcal{P}$ whose lifetime excursion length $\sigma(\zeta)$ is precisely equal to σ_0. We denote $\dot{\mathcal{P}}$ the set of (W,ζ) such that (ζ_s) and $(\hat{W}_s)$ are two absolutely continuous function and we use the notations $\dot{\zeta}_s$, $\dot{\hat{W}}_s$ to denote in this case the derivatives with respect to s. The space $\mathcal{P}$ is endowed with the metric

$$d((W,\zeta),(W',\zeta')) = |\sigma(\zeta)-\sigma(\zeta')| + \sup_{s\geq 0}|\zeta_s - \zeta'_s| + \sup_{s\geq 0}\sup_{r\leq \zeta_s\vee\zeta'_s}|W_s(r)-W'_s(r)|.$$

Lemma 9. *Let us call J the function on $\mathcal{P}$ defined by*

$$J(W) = \frac{1}{2}\int_0^\sigma \dot{\zeta}_s^2\, ds + \frac{1}{4}\int_0^\sigma \frac{|\dot{\hat{W}}_s|^2}{|\dot{\zeta}_s|} ds$$

if $(W,\zeta) \in \dot{\mathcal{P}}$ and $+\infty$ if not. Then,

1. *the function J is invariant under scaling :*

$$\forall \alpha > 0,\ \ J(\theta_\alpha(W)) = J(W); \tag{34}$$

2. *for all $L,\sigma_1 > 0$, the set $\{W;\ \ J(W) \leq L, \sigma(W) \leq \sigma_1\}$ is compact;*
3. *the function J is lower semi-continuous that is, for every sequence (W_n) converging to W_0 in $\mathcal{P}$,*

$$\liminf_{n\to+\infty} J(W_n) \geq J(W_0).$$

Proof. The first assertion amounts to trivial changes of variables in the definition of J. A consequence is that, for any non empty subset B of $(0,+\infty)$ the set $\{W;\ \ J(W) \in B\}$ is unbounded. The second point was obtained in [S1] when J is restricted to $\mathcal{P}^1$. We generalize easily using the first point and the continuity of the scaling operators (θ_α). The last assertion concerning the lower semi-continuity of J is then a direct corollary of the second one.

We want now to formulate a large deviation principle under $\mathbb{N}_0$. But it can happen that closed sets that were "large deviation events" under $\mathbb{N}_0^1$ are no longer of exponentially small probability. For instance, using the notation (6),

$$\mathbb{N}_0[R(W^r) \geq 1] = \mathbb{N}_0[r^{-3/4}\, R(W) \geq 1] = \frac{1}{r^{3/2}}\, \mathbb{N}_0[R(W) \geq 1]$$

where the last equality obtains by scaling as stated in Equation (15). It is much different under $\mathbb{N}_0^1$, where we have

$$\lim_{r\to+\infty} \frac{1}{r} \log \mathbb{N}_0^1\, [R(W^r) \geq 1] = -\frac{3}{2}$$

as proved in [S1]. Also, because we work under an infinite measure we will have to restrict the sets to which the large deviation principle applies.

Theorem 10. *The laws under $\mathbb{N}_0$ of W^r satisfy a large deviation result as $r \to +\infty$ with speed r and rate function J in the following way :*

- *for every open subset U of $\mathcal{P}$,*

$$\liminf_{r\uparrow+\infty} \frac{1}{r}\ \log \mathbb{N}_0\, [W^r \in U] \geq -\inf_U J \tag{35}$$

- *for every closed subset $F \subset \{W \in \mathcal{P};\ \sigma_1 \leq \sigma(W) \leq \sigma_2\}$ with $\sigma_2 > \sigma_1 > 0$,*

$$\limsup_{r\uparrow+\infty} \frac{1}{r}\ \log \mathbb{N}_0\, [W^r \in F] \leq -\inf_F J. \tag{36}$$

Proof. For A Borel subset of $\mathcal{P}$, we condition with respect to the length of the lifetime excursion and we note that the scaling operators (θ_α) defined by Formula (16) operate on $\mathcal{P}$ so that $\mathbb{N}_0^\sigma$ is the image of $\mathbb{N}_0^1$ by θ_σ. We obtain :

$$\mathbb{N}_0(W^r \in A) = \int_0^{+\infty} \frac{d\sigma}{2\sqrt{2\pi}\,\sigma^{3/2}}\ \mathbb{N}_0^1(\theta_\sigma(W^r) \in A). \tag{37}$$

By the contraction principle applied to the continuous function θ_σ and to the large deviation principle given in [S1], we have,

- for every open subset U of $\mathcal{P}$,

$$\liminf_{r\uparrow+\infty} \frac{1}{r}\ \log \mathbb{N}_0^1\, [\theta_\sigma(W^r) \in U] \geq -\tilde{I}^\sigma(U) \tag{38}$$

- for every closed subset K of $\mathcal{P}$,

$$\limsup_{r\uparrow+\infty} \frac{1}{r}\ \log \mathbb{N}_0^1\, [\theta_\sigma(W^r) \in K] \leq -\tilde{I}^\sigma(K) \tag{39}$$

where, for $A \subset \mathcal{P}$,

$$\tilde{I}^\sigma(A) = \inf\{\tilde{J}(\tilde{W});\ \tilde{W} \in \mathcal{P}^1 \text{ and } \theta_\sigma(\tilde{W}) \in A\} = \inf_{A\cap\mathcal{P}^\sigma} J.$$

For the last equality, note that if $\theta_\sigma(\tilde{W}) = W$, we know by (34) that $\tilde{J}(\tilde{W}) = J(W)$. As a consequence, for any $A \subset \mathcal{P}$, we have

$$\inf_{\sigma>0} \tilde{I}^\sigma(A) = \inf_{W\in A} J(W). \tag{40}$$

We now want to use Inequality (38) [respectively Inequality (39)] in Formula (37) to get, via Equation (40), the desired result (35) [respectively (36)]. This is a kind of Laplace method for the integral in (37).

We first prove (35). For shortness we set $\tilde{I} = \inf_{\sigma>0} \tilde{I}^\sigma(U) = \inf_U J$ and fix $\varepsilon > 0$. Let $\sigma_0 > 0$ be such that $\tilde{I}^{\sigma_0}(U) \le \tilde{I}+\varepsilon$. We can find $W_0 \in \mathcal{P}^1$ such that $\theta_{\sigma_0}(W_0) \in U$ and $\tilde{J}(W_0) \le \tilde{I}^{\sigma_0}(U)+\varepsilon$. Then, for σ in a neighbourhood of σ_0, say $\sigma \in]\sigma_0-\eta, \sigma_0+\eta[$, we have $\theta_\sigma(W_0) \in U$ because U is open. Therefore,

$$\tilde{I}^\sigma(U) \le \tilde{J}(W_0) \le \tilde{I} + 2\varepsilon. \tag{41}$$

But trivially,

$$\mathbb{N}_0(W^r \in U) \ge \int_{\sigma_0-\eta}^{\sigma_0+\eta} \frac{d\sigma}{2\sqrt{2\pi}\,\sigma^{3/2}}\ \mathbb{N}_0^1(\theta_\sigma(W^r) \in U).$$

Then

$$\begin{aligned}
&\liminf_{r\uparrow+\infty} \left\{\frac{1}{r}\ \log \mathbb{N}_0\left[W^r \in U\right] + (\tilde{I}+3\varepsilon)\right\}\\
&= \liminf_{r\uparrow+\infty}\ \log\left(\mathbb{N}_0\left[W^r \in U\right] e^{r(\tilde{I}+3\varepsilon)}\right)^{\frac{1}{r}}\\
&\ge \liminf_{r\uparrow+\infty}\ \log\left(\int_{\sigma_0-\eta}^{\sigma_0+\eta} \frac{d\sigma}{2\sqrt{2\pi}\,\sigma^{3/2}}\ \mathbb{N}_0^1(\theta_\sigma(W^r)\in U)e^{r(\tilde{I}+3\varepsilon)}\right)^{\frac{1}{r}}\\
&\ge 0
\end{aligned}$$

which proves (35) by letting ε tend to 0. The last inequality follows from Fatou's lemma

$$\begin{aligned}
&\liminf_{r\uparrow+\infty} \int_{\sigma_0-\eta}^{\sigma_0+\eta} \frac{d\sigma}{2\sqrt{2\pi}\,\sigma^{3/2}}\ \mathbb{N}_0^1(\theta_\sigma(W^r)\in U)e^{r(\tilde{I}+3\varepsilon)}\\
&\ge \int_{\sigma_0-\eta}^{\sigma_0+\eta} \frac{d\sigma}{2\sqrt{2\pi}\,\sigma^{3/2}}\ \liminf_{r\uparrow+\infty}\left\{\mathbb{N}_0^1(\theta_\sigma(W^r)\in U)e^{r(\tilde{I}+3\varepsilon)}\right\}
\end{aligned}$$

and

$$\liminf_{r\uparrow+\infty}\left\{\mathbb{N}_0^1(\theta_\sigma(W^r)\in U)e^{r(\tilde{I}+3\varepsilon)}\right\} = +\infty$$

because of (38) and (41).

We pass to the proof of (36) concerning the closed subset $F \subset \{W;\ \sigma_1 \leq \sigma(W) \leq \sigma_2\}$. We note that $\sigma(W^r) = \sigma(W)$ and condition with respect to σ, as in (37) :

$$\begin{aligned}\frac{1}{r}\,\log \mathbb{N}_0\,[W^r \in F] &= \frac{1}{r}\,\log \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{2\sqrt{2\pi}\,\sigma^{3/2}}\ \ \mathbb{N}_0^1(W^r \in \theta_\sigma^{-1}(F \cap \mathcal{P}^\sigma)) \\ &\leq \frac{1}{r}\,\log \left[\frac{\sigma_2 - \sigma_1}{2\sqrt{2\pi}\,(\sigma_1)^{3/2}} \mathbb{N}_0^1\left(W^r \in \tilde{F}\right)\right]\end{aligned}$$

where

$$\tilde{F} = \bigcup_{\sigma_1 \leq \sigma \leq \sigma_2} \theta_\sigma^{-1}(F \cap \mathcal{P}^\sigma).$$

By the continuity of $(\sigma, W) \to \theta_\sigma(W)$ and the closedness of F we obtain easily that $\tilde{F}$ is closed. Therefore we can apply (39) to claim that the limsup of the righthand side in the above inequality is lower than

$$\tilde{I}^1(\tilde{F}) = \inf_{\tilde{F}} \tilde{J} = \inf_F J.$$

This completes the proof of (36).

6. Large deviations for a surviving snake

We want to formulate a large deviation result for the Brownian snake, conditioned as usual to survive the killing at constant rate V, up to time τ_1 and with Brownian spatial motion. We will now consider the following rescaling :

$$W^{[r]} = \left(\frac{1}{r}\,W_{rs}(r\,\cdot),\ \frac{1}{r}\zeta_{rs}\right)_{s \geq 0}$$

because, for any measurable subset A of $\mathcal{P}$, the scaling property (15) implies that

$$\begin{aligned}\mathbb{N}_0^{(V)}\left[W^{[r]} \in A\right] &= \mathbb{N}_0\left[\mathbf{1}_{\{W^{[r]} \in A\}}\ e^{-V\,\sigma(W)}\right] \\ &= \frac{1}{\sqrt{r}}\ \mathbb{N}_0\left[\mathbf{1}_{\{\theta_r(W)^{[r]} \in A\}}\ e^{-V\,\sigma(\theta_r(W))}\right] \\ &= \frac{1}{\sqrt{r}}\ \mathbb{N}_0\left[\mathbf{1}_{\{W^r \in A\}}\ e^{-V\,r\,\sigma(W^r)}\right]. \end{aligned} \tag{42}$$

We have used

$$\sigma(\theta_r(W)) = r\,\sigma(W) = r\,\sigma(W^r) \text{ and } \theta_r(W)^{[r]} = W^r\,.$$

On the expression (42), we would like to argue as in the Varadhan-Laplace Lemma, using the large deviation principle for W^r given in Theorem 10. However, we will have to impose, as in Theorem 10, a restriction on the closed sets K for which the result holds. More precisely, it will be the existence of $p > 1$ such that

$$\int_0^{+\infty} \frac{d\sigma}{2\sqrt{2\pi}\,\sigma^{3/2}}\ e^{-r\,V\,\sigma}\ \mathbb{N}_0^\sigma(W^r \in K)^{1/p} \tag{43}$$

stays bounded when $r \to +\infty$. This technical condition is only a little stronger than the boundedness of

$$\int_0^{+\infty} \frac{d\sigma}{2\sqrt{2\pi}\,\sigma^{3/2}}\, e^{-r\,V\,\sigma}\, \mathbb{N}_0^{\sigma}(W^r \in K) = \sqrt{r}\, \mathbb{N}_0^{(V)}\left[W^{[r]} \in K\right]$$

which is obviously necessary to give sense to the following result.

Theorem 11. *The laws under $\mathbb{N}_0^{(V)}(dW)$ of $W^{[r]}$ satisfy as $r \to +\infty$ a large deviation principle with rate function $W \to J(W) + V\,\sigma(W)$ in the following way :*

- *for every open subset U of $\mathcal{K}$,*

$$\liminf_{r\uparrow+\infty} \frac{1}{r}\, \log \mathbb{N}_0^{(V)}\left[W^{[r]} \in U\right] \geq -\inf_U (J + V\,\sigma) \tag{44}$$

- *for every closed subset K of $\mathcal{K}$ satisfying the assumption (43),*

$$\limsup_{r\uparrow+\infty} \frac{1}{r}\, \log \mathbb{N}_0^{(V)}\left[W^{[r]} \in K\right] \leq -\inf_K (J + V\,\sigma).$$

Proof. As the factor $1/\sqrt{r}$ does not interfere with exponential speed, we can prove the result for the last quantity appearing in (42) instead of $\mathbb{N}_0^{(V)}\left[W^{[r]} \in A\right]$, in both cases $A = U$ open and $A = K$ closed.

We first let U be an open subset of $\mathcal{P}$, $y \neq 0$ belong to U and $\eta > 0$ be such that $B(y,\eta) \subset U$. Then $\sigma(W) \leq \sigma(y) + \eta$ if W belongs to $B(y,\eta)$. Hence,

$$\begin{aligned}
\mathbb{N}_0\left[\mathbf{1}_{\{W^r\in U\}}\, \exp\left(-r\; V\; \sigma(W^r)\right)\right] &\geq \mathbb{N}_0\left[W^r \in U\right]\, \exp -(r(\sigma(y)+\eta)) \\
&\geq \mathbb{N}_0\left[W^r \in B(y,\eta)\right]\, \exp -(r(\sigma(y)+\eta)).
\end{aligned}$$

Therefore, using Theorem 10 we get

$$\begin{aligned}
&\liminf_{r\uparrow+\infty} \frac{1}{r} \log \mathbb{N}_0\left[\mathbf{1}_{\{S(W^r)\in U\}}\, \exp\left(-r\; V\; \sigma(W^r)\right)\right] \\
&\quad\geq\; -\inf_{B(y,\eta)} J - (\sigma(y)+\eta) \\
&\quad\geq\; -J(y) - \sigma(y) - \eta.
\end{aligned}$$

We let $\eta \downarrow 0$ and maximize over $y \in U$ to get (44).

Now we consider a closed subset K of $\mathcal{P}$ satisfying (43). We may suppose that $0 \notin K$. We set, for $L, \sigma_1 > 0$,

$$\begin{aligned}
K_1 &= K \cap \{W;\; J(W) \leq L\} \cap \{\sigma \leq \sigma_1\}, \\
K_2 &= K \cap \{W;\; J(W) \leq L\} \cap \{\sigma \geq \sigma_1\}, \\
K_3 &= K \cap \{W;\; J(W) > L\}.
\end{aligned}$$

We know by Lemma 9 that $\{W;\; J(W) \leq L\} \cap \{\sigma \leq \sigma_1\}$ is compact so that K_1 can be covered by a finite number of closed balls of arbitrary small radius η :

$$K_1 \subset \bigcup_{i=1}^{N} \overline{B}(y_i,\eta).$$

Then

$$\begin{aligned}
&\mathbb{N}_0\left[\mathbf{1}_{\{W^r\in K_1\}}e^{-r\,V\,\sigma(W^r)}\right]\\
&\le \sum_{i\le N}\mathbb{N}_0\left[\mathbf{1}_{\{W^r\in\overline{B}(y_i,\eta)\}}\;\; e^{-r\,V\,\sigma(W^r)}\right]\\
&\le \sum_{i\le N}\mathbb{N}_0\left[W^r\in\overline{B}(y_i,\eta)\right]\;\; e^{-r\,V\,(\sigma(y_i)-\eta)}\quad .
\end{aligned}$$

We deduce, using Theorem 10,

$$\begin{aligned}
&\limsup_{r\uparrow+\infty}\frac{1}{r}\log\mathbb{N}_0\left[\mathbf{1}_{\{W^r\in K_1\}}\;\exp(-r\,V\,\sigma(W^r))\right]\\
&\le\quad -\inf_{i\le N}\left(\inf_{\overline{B}(x_i,\eta)}J-V\,(\sigma(y_i)-\eta)\right)\\
&\le\quad -\inf_{i\le N}\inf_{\overline{B}(x_i,\eta)}(J+V\,\sigma)\;+2V\eta\\
&=\quad -\inf\left\{(J+V\sigma)(W);\;W\in\bigcup_{i\le N}\overline{B}(x_i,\eta)\right\}\;+2V\eta\\
&\overset{\eta\to0}{\longrightarrow}\quad -\inf_{K_1}(J+V\sigma)
\end{aligned}\tag{45}$$

For the last convergence we have used the lower semi-continuity of J and the continuity of $W\to\sigma(W)$.

Concerning the contribution of K_2, we have :

$$\mathbb{N}_0\left[\mathbf{1}_{\{W^r\in K_2\}}e^{-r\,V\,\sigma(W^r)}\right]\le e^{-r\,\sigma_1 V}\;\;\mathbb{N}_0(\sigma\ge\sigma_1).$$

We deduce

$$\limsup_{r\uparrow+\infty}\frac{1}{r}\log\mathbb{N}_0\left[\mathbf{1}_{\{W^r\in K_2\}}\;\exp -r\,V\,\sigma(W^r)\right]\le-\sigma_1\,V\,. \tag{46}$$

Finally, concerning the contribution of K_3,

$$\begin{aligned}
&\mathbb{N}_0\left[\mathbf{1}_{\{W^r\in K_3\}}e^{-r\,V\,\sigma(W^r)}\right]\\
&\le\;\int_0^{+\infty}\frac{e^{-r\,V\,\sigma}\;d\sigma}{\sqrt{2\pi}\,\sigma^{3/2}}\;\;\mathbb{N}_0^\sigma(W^r\in K;\;J(W^r)>L)\\
&\le\;\int_0^{+\infty}\frac{e^{-r\,V\,\sigma}\;d\sigma}{\sqrt{2\pi}\,\sigma^{3/2}}\;\;\mathbb{N}_0^\sigma(W^r\in K)^{1/p}\;\;\mathbb{N}_0^\sigma(J(W^r)>L)^{1/q}\\
&\le\;\left(\int_0^{+\infty}\frac{e^{-r\,V\,\sigma}\;d\sigma}{\sqrt{2\pi}\,\sigma^{3/2}}\;\mathbb{N}_0^\sigma(W^r\in K)^{1/p}\right)\;\;\mathbb{N}_0^1(W^r\in\{J>L\})^{1/q}\;.
\end{aligned}\tag{47}$$

We have used as before the conditioning of $\mathbb{N}_0$ according to excursion length, then the Hölder inequality, where q is the conjugate exponent of p i.e. $(1/p)+$

$(1/q) = 1$. Then, we have noted that

$$\mathbb{N}_0^{\sigma}\left(J(W^r) > L\right) = \mathbb{N}_0^{1}\left(J(\theta_\sigma(W^r)) > L\right) = \mathbb{N}_0^{1}\left(J(W^r) > L\right)$$

because of the invariance of J by scaling, as proved in Lemma 9. Recalling that the quantity in brackets in (47) is by the assumption (43) supposed to be bounded, we obtain

$$\limsup_{r\uparrow+\infty} \frac{1}{r} \log \mathbb{N}_0 \left[\mathbf{1}_{\{W^r \in K_3\}} \; \exp(-r\, V\, \sigma(W^r))\right] \leq -\frac{L}{2\,q}\,. \tag{48}$$

The combination of Equations (45), (46), (48) completes the proof of the Theorem as we recall that L, σ_1 are arbitrary.

7. Applications

The following propositions give, when the killing is constant equal to V, large deviation estimates for the exit from a big ball and the extinction time.

Proposition 12. $R = \sup_{s\leq\sigma} |\hat{W}_s|$ *satisfies*

$$\lim_{r\uparrow+\infty} \frac{1}{r} \log \mathbb{N}_0^{(V)}\left[R \geq r\right] = -2^{5/4}\, V^{1/4}.$$

Proof. We apply Theorem 11 to

$$U = \{W \in \mathcal{P};\ R(W) > 1\} \text{ and } K = \{W \in \mathcal{P};\ R(W) \geq 1\}$$

which are respectively open and closed in $\mathcal{P}$. This set K satisfies the assumption (43) because, as obtained in [S1],

$$\mathbb{N}_0^{\sigma}(W^r \in K) = \mathbb{N}_0^{1}\left(\sigma^{1/4}\, R(W^r) \geq 1\right) \leq e^{-c\,\frac{r}{\sigma^{1/3}}}.$$

The result will be proved if we show that

$$\begin{aligned} \inf\{J(W) + V\,\sigma(W);\ R(W) \geq 1\} &= \inf\{J(W) + V\,\sigma(W);\ R(W) > 1\} \\ &= 2^{5/4}\, V^{1/4}. \end{aligned}$$

In the search of the first infimum (second one is similar) we can restrict ourselves to (W_s) of the type

$$\zeta_s = \frac{2sh}{\sigma}\, \mathbf{1}_{[0,\frac{\sigma}{2}]} + 2h\left(1 - \frac{s}{\sigma}\right)\, \mathbf{1}_{(\frac{\sigma}{2},\sigma]}$$

$$W_s = \frac{2s}{\sigma}\, u\, \mathbf{1}_{[0,\frac{\sigma}{2}]} + 2h\left(1 - \frac{s}{\sigma}\right)\, u\, \mathbf{1}_{(\frac{\sigma}{2},\sigma]}$$

where $u \in \mathbb{R}^d$ has norm equal to 1 and for such a (W_s) we have

$$J(W) + V\,\sigma(W) = \frac{2h^2}{\sigma} + \frac{1}{2h} + V\,\sigma. \tag{49}$$

Indeed the form of the rate function shows that we can restrict ourselves to snakes that are symmetric with respect to $\sigma/2$. Then the Cauchy-Schwarz inequality gives

$$\begin{aligned} J(W) + V\,\sigma(W) &= \int_0^{\sigma/2} \dot\zeta_s^2\, ds + \frac{1}{2}\int_0^{\sigma/2} \frac{|\dot{\hat W}_s|^2}{|\dot\zeta_s|} ds + V\,\sigma \\ &\geq \frac{2}{\sigma}\left(\int_0^{\sigma/2} \dot\zeta_s\, ds\right)^2 + \frac{1}{2}\frac{\left(\int_0^{\sigma/2} |\dot{\hat W}_s|\, ds\right)^2}{\int_0^{\sigma/2} |\dot\zeta_s|\, ds} + V\,\sigma \\ &\geq \frac{2\,\zeta_{\sigma/2}^2}{\sigma} + \frac{|\hat W_{\sigma/2}|^2}{2\,\zeta_{\sigma/2}} + V\,\sigma. \end{aligned}$$

Standard optimization of (49) gives the optimal values

$$h = 2^{-5/4}\, V^{-1/4}, \quad \sigma = 2^{-3/4}\, V^{-3/4}$$

for which $J(W) + V\,\sigma(W) = 2^{5/4}\, V^{1/4}$.

Proposition 13. *$H = \sup_{s\leq\sigma} |\zeta_s|$ satisfies*

$$\lim_{r\uparrow+\infty} \frac{1}{r} \log \mathbb{N}_0^{(V)}\,[H \geq r] = -2^{3/2}\,\sqrt{V}.$$

Proof. In this setting, checking (43) follows from

$$\mathbb{N}_0^{\sigma}(H(W^r) \geq 1) \leq e^{-c\,\frac{r}{\sigma}}.$$

The present proof amounts to show that

$$\begin{aligned} \inf\{J(W) + V\,\sigma(W);\ H \geq 1\} &= \inf\{J(W) + V\,\sigma(W);\ H > 1\} \\ &= 2^{3/2}\,\sqrt{V}. \end{aligned} \tag{50}$$

The derivation of (50) is done as in the proof of the previous proposition reducing to easy optimization on real variables.

8. De-conditioning

We have seen in the previous sections that conditioning a Brownian snake to survive a killing procedure leads to a b-snake i.e. a Brownian snake with drift. Such a b–snake can also be obtained by other methods, as proved in [AS]. In the following proposition we are going to see that a b–snake can be transformed into a (standard) Brownian snake by "inserting small snakes".

Proposition 14. *Let (W_s) be a b–snake where $b : \mathcal{W} \to \mathbb{R}_+$ is continuous. Suppose that, conditionally on (W_s), the point measure Λ on $\mathbb{R}_+ \times \mathcal{W}$ is a Poisson measure with intensity*

$$ds\ 2\ b(W_s)\ \mathbb{N}_{\hat W_s}(dW). \tag{51}$$

The increasing right-continuous function

$$A_r = r + \int_{\{s\leq r\}} \Lambda(ds\, dW)\ \sigma(W)$$

has a jump at time r of height $A_r - A_{r-} = \sigma(W')$ if $\Lambda(\{(r,W')\}) \neq 0$. We define the $\mathcal{W}$–valued process W^1 by $W^1_s = W_r$ if $A_r = s$ and

$$W^1_s = W_r \oplus W'_{s-A_{r-}} \text{ if } A_{r-} \leq s < A_r \text{ and } \Lambda(\{(r,W')\}) \neq 0.$$

Then W^1 is a (standard) Brownian snake.

Proof. The meaning of this Proposition is that a b–snake can be transformed into a (standard) Brownian snake by inserting "snake excursions" W^i, $i \in I$ at respective times s_i so that in particular the lifetime of the inserted snake ζ^i realizes a Brownian excursion above level ζ_{s_i}, the measure

$$\sum_{i\in I} \delta_{(s_i,W^i)}$$

being a Poisson measure with intensity given by (51). Restricted to the lifetime process, this Proposition amounts to a result on Brownian motion given in [S4]. The proofs given there can be easily extended to path–valued processes to obtain this statement.

We now reformulate it in terms of super-Brownian motion or even historical Brownian motion. We introduce the infinite measure $\mathbb{Q}_{t,x}$ on $C(\mathbb{R}_+, \mathcal{M}_F(\mathcal{W}))$ which is the "law" of the historical process associated to a Brownian snake distributed according to the excursion measure from x, and starting at time t : under $\mathbb{Q}_{t,x}(dY)$ we have $Y_u = 0$ for $u \leq t$ and

$$\int \mathbb{Q}_{t,x}(dY)\; F(Y_{t+u}, u \geq 0) = \int \mathbb{N}_x(dW) F\left(H_u(W),\; u \geq 0\right)$$

where, on the right-hand side, $H_u(W)$ is defined as in (10).

Theorem 15. *Let (H_t) be an historical Brownian motion with drift b where $b : \mathcal{W} \to \mathbb{R}_+$ is continuous. Suppose that, conditionally on (H_t), the point measure Θ on $C(\mathbb{R}_+, \mathcal{M}_F(\mathcal{W}))$ is a Poisson measure with intensity*

$$\int_0^{+\infty} dt\; H_t(dw)\; b(w)\; \mathbb{Q}_{t,\hat{w}} \tag{52}$$

Then

$$\left(H_t + \int H'_t\; \Theta(dH')\right)_{t\geq 0}$$

is an historical Brownian motion (without drift).

Proof. We suppose that the initial measure H_0 is $\delta_{\tilde{x}_0}$ for $x_0 \in \mathbb{R}^d$. The general case consists in taking a Poisson measure of such processes (see [Lg] p.61-64) and works as well. Using the notations defined at the begining of the proof of the previous Proposition, it follows from this Proposition that the historical Brownian motion (H^1_t) associated to the Brownian snake W^1 can be written

$$H^1_{\cdot} = H_{\cdot} + \sum_{i\in I} H^i_{\cdot - \zeta_{s_i}}$$

where H [resp. H^i] is relative to W [resp. W^i]. We use the convention $H^i_u = 0$ for $u < 0$. Thus the Proposition will be prouved once we establish that the point measure

$$\sum_{i\in I} \delta_{H^i_{\cdot-\zeta_{s_i}}}$$

is a Poisson point measure with intensity given by (52). But it follows from (51) that this intensity Θ is such that

$$\begin{aligned}\int \Theta(dY)\, \psi(Y) &= 2\int_0^{\tau_1} ds\, b(W_s)\, \mathbb{N}_{\hat{W}_s}(dW)\, \psi\left(H_{\cdot-\zeta_s}(W)\right)\\ &= 2\int_0^{\infty} dt \int H_t(dw)\, b(w)\, \mathbb{N}_{\hat{w}}(dW)\, \psi\left(H_{\cdot-t}(W)\right)\end{aligned}$$

where the last equality follows from (12). We obtain the sought-after result.

9. Everlasting super-Brownian motion with drift

9.1. **Remarks on the case without drift.** Let us from now on denote $\mathbb{P}_\mu$ the law of the (ordinary) super-Brownian motion (Y_t) starting from μ, a finite measure on $\mathbb{R}^d$. Since $(Y_t(1))$ is a martingale under $\mathbb{P}_\mu$, a new law of measure valued process $\tilde{\mathbb{P}}_\mu$ may be defined by

$$\tilde{\mathbb{E}}_\mu\left[F(Y_s,\ s\le t)\right] = \mathbb{E}_\mu\left[F(Y_s,\ s\le t)\ \frac{Y_t(1)}{\mu(1)}\right].$$

It is called the law of super-Brownian motion conditioned to non-extinction because we have the following property, for bounded continuous F,

$$\tilde{\mathbb{E}}_\mu\left[F(Y_s,\ s\le t)\right] = \lim_{h\to+\infty} \mathbb{E}_\mu\left[F(Y_s,\ s\le t)\ \middle| Y_{t+h}(1) > 0\right]$$

which is easy to obtain using the asymptotics of the extinction probability. See for instance [Ev] or [Ov] for more on this conditioning procedure and the resulting process. We claim that $\tilde{\mathbb{P}}_\mu$ is also the law of super-Brownian motion conditioned to have an infinite total mass $Z = \int_0^{+\infty} Y_t(1)\, dt$, in the following way.

Proposition 16. *For $F = F(Y_s,\ s\le t)$ continuous and bounded, we have :*

$$\tilde{\mathbb{E}}_\mu[F] = \lim_{L\to+\infty} \mathbb{E}_\mu\left[F | Z = L\right]$$

The fact that the two conditionings either by infinite extinction time or by infinite total mass lead to the same limit has been observed long ago in the case of a critical Galton-Watson process by Kennedy. In the present setting we provide a

Sketch of proof. It is proved in [S3] that the laws $\mathbb{P}_\mu\left[\cdot | Z = L\right]$ are the solutions of the martingale problem (stopped at extinction)

$$\begin{cases} \forall \phi \in \mathcal{D}(A), \;\; Y_t(\phi) = \mu(\phi) + \int_0^t Y_s(A\phi)\, ds \\ \qquad\qquad + \int_0^t \left(\frac{4}{Y_s(1)} - \frac{Y_s(1)}{L - \int_0^s Y_r(1)\, dr} \right) \; Y_s(\phi)\, ds + M_t(\phi) \\ \text{where the local martingale } M \text{ has quadratic variation} \\ \langle M(\phi) \rangle_t = 4 \int_0^t Y_s(\phi^2)\, ds. \end{cases}$$

Note that the factor 4 has to be added here because we have chosen to work with super-Brownian motion with branching rate 4. A reformulation is that the law $\mathbb{P}_\mu\left[\cdot|Z = L\right]$ has a density with respect to $\mathbb{P}_\mu$ of the form

$$\begin{aligned} & \exp\left[\int_0^t \left(\frac{1}{Y_s(1)} - \frac{Y_s(1)}{4L - 4\int_0^s Y_r(1)\, dr} \right) dY_s(1) \right. \\ & \left. \quad - \;\; 2\int_0^t \left(\frac{1}{Y_s(1)} - \frac{Y_s(1)}{4L - 4\int_0^s Y_r(1)\, dr} \right)^2 \; Y_s(1)\, ds \right] \end{aligned} \tag{53}$$

when we restrict to events $\sigma(Y_s;\;\; s \leq t)$–measurable and contained in $\{Y;\ \forall s \leq t,\ M \geq Y_s(1) \geq (1/M)\}$ for a certain $M > 0$. This localization is meant to ensure the existence of the density (53), when L is large enough. Passing to the limit $L \to +\infty$ in (53) we obtain the limit (in probability)

$$\exp\left[\int_0^t \frac{1}{Y_s(1)} dY_s(1) - 2\int_0^t \frac{1}{Y_s(1)}\; ds \right]$$

but this is equal to $Y_t(1)/Y_0(1)$ by Itô formula applied to $\log Y_t(1)$. We get precisely the density of $\tilde{\mathbb{P}}_\mu$ with respect to $\mathbb{P}_\mu$. Thus we have obtained on the localization set the convergence of the densities which implies weak convergence by the so-called Schéffé Lemma. The proof may be completed by checking the technical argument that the localization set has a mass under the laws $\tilde{\mathbb{P}}_\mu$ and $\mathbb{P}_\mu\left[\cdot|Z = L\right]$, L large, (restricted to $\sigma(Y_s, s \leq t)$) which is arbitarily uniformly close to 1 when M is chosen large enough.

9.2. **Case of constant drift.** We now consider the law $\mathbb{P}^b_\mu$ of super-Brownian motion with constant drift $-b$ starting from μ a finite measure on $\mathbb{R}^d$. We note that

$$\tilde{\mathbb{E}}^b_\mu\left[F(Y_s,\ s \leq t)\right] = \mathbb{E}_\mu\left[F(Y_s,\ s \leq t)\ \exp\left(-\frac{b}{2}(Y_t(1) - \mu(1)) - \frac{b^2}{2}\int_0^t Y_s(1)\, ds \right)\right] \tag{54}$$

We want to condition this process not to die following the lines of the undrifted case. We first state a lemma which recalls “well-known” facts. However the formulation may differ from other sources because our super-Brownian motion has branching rate 4.

Lemma 17. *Let $X_t = Y_t(1)$ be the “mass process” of a super-Brownian motion (without drift) (Y_t), starting from μ.*
(i) (X_t) is a Feller diffusion with generator $2x\,\frac{d^2}{dx^2}$, that is a squared Bessel

process of dimension 0, starting from $\mu(1)$.
(ii) For $\gamma > 0$,
(55)

$$\mathbb{E}^b_\mu\left[\exp\left(-\gamma\, X_t - \frac{b^2}{2}\int_0^t X_s\, ds\right)\right] = \exp - \left(\frac{\mu(1)b}{2}\,\frac{b\,\sinh(bt) + 2\gamma\,\cosh(bt)}{b\,\cosh(bt) + 2\gamma\,\sinh(bt)}\right).$$

Let us now suppose that (Y_t) *is under* $\mathbb{P}^b_\mu$ *a super-Brownian motion with constant drift* $-b$ *starting from* μ *and, as before,* $X_t = Y_t(1)$ *having initial value* $X_0 = \mu(1)$.
(iii) We have, when $t \to +\infty$,

$$\mathbb{P}^b_\mu\,[X_t > 0] \sim \mu(1)\; b\;\exp(-2bt)\,. \tag{56}$$

(iv) $\left(\dfrac{X_t\;\exp(2bt)}{\mu(1)}\right)$ *is a martingale under* $\mathbb{P}^b_\mu$.

Proof. For (i) see the martingale problem. Then (ii) is a classical result cf [RY] Theorem XI.1.7. Taking into account the density of $\mathbb{P}^b_\mu$ with respect to $\mathbb{P}_\mu$ as given in (54), the right-hand side of (55) gives, for $\gamma = \lambda + (b/2)$, the value of $\mathbb{E}^b_\mu[\exp(-\lambda\, X_t)]$. Passing to the limit $\lambda \to +\infty$ we obtain

$$\mathbb{P}^b_\mu\,[X_t > 0] = 1 - \exp - \left(\frac{\mu(1)b}{2}\;\frac{\cosh(bt) - \sinh(bt)}{\sinh(bt)}\right)$$

hence (iii). Finally (iv) is checked using (54) and (55).

We define the law $\tilde{\mathbb{P}}^b_\mu$ of the everlasting super-Brownian motion with drift $-b$ using the martingale found in (iv) of the above Lemma i.e.

$$\tilde{\mathbb{E}}^b_\mu\,[F] = \mathbb{E}^b_\mu\left[F\;\frac{X_t\;\exp(2bt)}{\mu(1)}\right]$$

for all measurable $F = F(Y_s,\; s \le t)$, for instance bounded. Then point (iii) of the above Lemma shows that, for F as before,

$$\tilde{\mathbb{E}}^b_\mu\,[F] = \lim_{h\to+\infty}\mathbb{E}^b_\mu\left[F\;\middle|\,Y_{t+h}(1) > 0\right]$$

so that the defined law is to be interpreted as the law of super-Brownian motion with drift conditioned not to die, as in the undrifted case. But the density of $\mathbb{P}^b_\mu$ with respect to $\mathbb{P}_\mu$ can also be written under the simple form

$$\frac{d\mathbb{P}^b_\mu}{d\mathbb{P}_\mu} = \exp\left(\frac{\mu(1)\,b}{2} - \frac{b^2}{2}\,Z\right)\;\text{ where } Z = \int_0^{+\infty} Y_t\;dt$$

as previously defined. Since this density is measurable with respect to Z, the conditional law of $\mathbb{P}^b_\mu$ knowing $Z = L$ is equal to the conditional law of $\mathbb{P}_\mu$ knowing $Z = L$ and as a consequence, has the same limit when $L \to +\infty$ which exists by Proposition 16. Thus we remark that conditioning the super-Brownian motion with drift to have infinite mass leads to the super-Brownian motion (without drift) conditioned to have infinite mass. This latter process was claimed in Proposition 16 to have the law $\tilde{\mathbb{P}}_\mu$ of super-Brownian motion (without drift) conditioned to non-extinction and that law is different from the law $\tilde{\mathbb{P}}^b_\mu$ of super-Brownian motion with drift conditioned to non-extinction.

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Laboratoire de Mathématiques, Université Blaise Pascal, Campus Universitaire des Cézeaux, 63177 Aubière cedex, France,

E-mail address: Laurent.Serlet@math.univ-bpclermont.fr